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Characterizations of Multivariate Differences and Associated Exponential Splines[1]

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The subject of this investigation is the class of difference functionals—linear combinations of finitely many function and/or derivative evaluations—which annihilate the nullspace of a certain constant coefficient differential operator. Any such functional can be viewed as an integral-differential operator whose Peano kernel is a compactly supported exponential spline. Besides extending some earlier results (1996, T. Kunkle, *J. Approx. Theory* **84**, 290–314), we show that these functionals are the only ones whose convolutions with the associated exponential truncated powers have compact support. It is then proven that, in case the functional depends entirely on function values at rational points, it must be a linear combination of forward differences. These results have applications in the areas of (a) placing compactly supported exponential splines in the span of the box splines, and (b) interpolation by exponential polynomials to function values at the support points of the forward difference functional. © 2000 Academic Press

Key Words: exponentials; polynomials; multivariate splines; box splines; exponential box splines; piecewise exponential.

1. INTRODUCTION

An earlier investigation of multivariate divided differences [15] focused on linear combinations of finitely many function evaluations whose nullspace contains that of a certain homogeneous differential operator, specifically, a product of directional differentiations. In that paper, it is shown that the convolution of such a functional with the corresponding truncated power is always compactly supported and that, like the B-spline, the resulting spline acts as a Peano kernel of the difference functional.

Examples of splines resulting from such a convolution include the tensor product B-spline, the box spline, and a box-like spline that later became

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the subject of two papers [16, 17]. But, because the difference functional was allowed to depend on function values only, none of the original analysis [15] applied to the simplex spline or to the spline obtained by letting the knots of the box-like spline coalesce.

Instead, such splines and their associated differences are the subject of this paper, which broadens the scope of investigation by allowing the difference functionals to consist of function and derivative evaluations and by allowing the differential operator to be nonhomogeneous, in which case the resulting splines are piecewise exponential-polynomial. Some of the results contained here (Section 3) are extensions of previous results to this more general setting. Others (Theorem 3.14, Sections 4 and 5) have no antecedents in the earlier paper but use these extensions as their foundation.

Theorems 3.12 and 3.14 and Corollary 3.13 concern a large class of linear functionals which include these differences. These basic results concern the convolution of such a functional with the corresponding truncated power, specifically its support and its role as the functional's Peano kernel. Theorem 3.15 characterizes differences in terms of their restrictions to lower dimensional subspaces. Corollary 3.16 states a geometric condition that is necessary in order for a nontrivial difference to consist of function evaluations only. Corollary 3.25 draws a connection between these differences and exponential polynomial interpolation.

The next results deal with spanning sets for the special class of differences which depend on function values only. Theorem 4.1 generalizes an old result from the polynomial case to the exponential polynomial case: when the directions of differences are linearly independent, then any such difference can be written entirely in terms of tensor product divided differences. Theorem 4.11 and Corollary 4.13 deal with the more difficult case that the directions of the difference are dependent. In that case, if the difference is supported on the rationals, then it must be a linear combination of the evenly spaced differences associated to the box spline.

The paper concludes with applications of these results and two open questions. Corollary 5.1 states that if a linear combination of rational shifts of the truncated power has compact support, then it must be a linear combination of box splines. Corollary 5.4 shows that, under some restrictions, one can interpolate to function values at the support points of the forward difference with exponential polynomials annihilated by the associated differential operator if and only if the forward difference of the data is zero.

There is a thorough review of univariate exponential divided differences and B-splines in Ref. [25] and a brief review in an earlier paper [17]. The (multivariate) exponential box spline has been studied by many people, including Ben-Artzi [1], de Boor [4], Dahmen and Micchelli [7], Dyn [8–10], Goodman and Taani [12], and Jia [13], Sivakumar [24, 26], and its inventor, Ron [21–24].

We begin by introducing some notation in the next section.

2. NOTATION

The ith component of a point x in $\mathbb{R}^d$ is denoted $x(i)$ and the scalar product between x and y, points in $\mathbb{R}^d$, is written $x \cdot y$. For $H \subset \mathbb{R}^d$, define $H^\perp$ to be the set of all vectors in $\mathbb{R}^d$ perpendicular to H in the usual sense.

The space of all d-variate polynomials is denoted Π and the space of exponential-polynomials (sums of exponentials-times-polynomials) is denoted $\operatorname{Exp} \Pi$. For $p \in \Pi$, the associated constant coefficient differential operator is denoted $p(D)$. By *test function* we mean an element of $C_c^\infty(\mathbb{R}^d)$, the space of all compactly supported infinitely differentiable functions on $\mathbb{R}^d$. For μ in $\mathbb{C}^d$, the function e_μ is given by the rule

$$e_\mu: \mathbb{R}^d \to \mathbb{R}: x \mapsto e^{\mu \cdot x}.$$

For x and y in $\mathbb{R}^d$, let $[x..y]$ be the set of all u for which $x \leqslant u \leqslant y$ in the usual sense. In case x and y are in $\mathbb{Z}^d$, let $\{x..y\}$ denote all the multi-integers in $[x..y]$.

The symbol S shall always stand for a subset of $\mathbb{R}^d$ with finite cardinality.

The cone generated by the subset T of $\mathbb{R}^d$ is denoted $[\![T]\!]_+$.

From the box-spline literature [3] we borrow the following convention. If N is a matrix with typical column v in $\mathbb{R}^d$, then one can think of N as a multiset in $\mathbb{R}^d$, eliminating the need for an index set for N other than N itself. For instance, $\mathbb{R}^{\mathrm{N}}$ denotes the space of all functions from N into $\mathbb{R}$, that is, the set of all real vectors indexed by N. When N is viewed as a map, this space is its domain:

$$\mathrm{N}: \mathbb{R}^{\mathrm{N}} \to \mathbb{R}^d: x \mapsto \mathrm{N}x := \sum_{v \in \mathrm{N}} x(v)\, v.$$

The image of this map (the column space of N) is denoted $\operatorname{ran} \mathrm{N}$. For $x \in \mathbb{R}^{\mathrm{N}}$ and $\mathrm{H} \subset \mathrm{N}$, let $x(\mathrm{H})$ denote the restriction of x to H.

For any set X, the vector in $\mathbb{R}^X$ of all 1s is written $\mathbb{1}$. The set X can often be made clear by context. For instance $A\mathbb{1}$ always stands for the sum of the columns of A.

Let $\alpha(v)$ denote the multiplicity of v in N; that is, the number of columns in N that are identical to v, so that $\alpha(v) > 0$ for all v in N. Let $v^{\alpha(v)}$ be the multisubset of N consisting of $\alpha(v)$ copies of v.

The differentiation operator in the (not necessarily unit) direction v is written D_v. If $\mu \in \mathbb{C}^N$, let

$$D_{v,\mu} = D_v + \mu(v)$$

and, more generally, for $\mathrm{H} \subseteq \mathrm{N}$,

$$D_{\mathrm{H},\mu} = \prod_{v \in \mathrm{H}} D_{v,\mu}.$$

Note that $D_{\mathrm{H},\mu}$ and $D_{\mathrm{H},\mu(\mathrm{H})}$ are the same operator. The set of all exponential-polynomials p for which $D_{\mathrm{N},\mu} p = 0$ is denoted Exp $\Pi_{\mathrm{N},\mu}$.

If $F(x \mid P)$ denotes the value at x of a function F depending on parameters P, the function itself $F(\cdot \mid P)$ will be written simply as $F(P)$.

Since the difference functionals of interest here are continuous on the space of test functions, we will draw no distinction between the functionals and the distributions which represent them. The support of a functional–distribution λ is denoted supp λ. The convolution of two functionals λ_1 and λ_2, if it exists, is defined by the rule

$$(\lambda_1 * \lambda_2)\, f = \lambda_1 \lambda_2 f(x_1 + x_2),$$

where λ_i views $f(x_1 + x_2)$ as a function of x_i.

3. BASIC RESULTS

The basic objects of study in this paper are the following classes of distributions.

DEFINITION 3.1. A *difference functional* is a linear combination of finitely many shifts of the Dirac δ and its derivatives, that is, a functional of the form

$$\lambda\colon C^\infty(\mathbb{R}^d) \to \mathbb{R}\colon f \mapsto \sum_S p_s(D)\, f(s), \tag{3.2}$$

where S is a finite subset of $\mathbb{R}^d$ and p_s is a d-variate polynomial for each s in S.

DEFINITION 3.3. Let N be a matrix whose columns lie in $\mathbb{R}^d \backslash \{0\}$ and let μ be in $\mathbb{C}^{\mathrm{N}}$. We say that the compactly supported distribution λ is a (N, μ)-*annihilator* if there is an open disk U containing the support of λ such that $f \in C^\infty(\mathbb{R}^d)$ and $D_{\mathrm{N},\mu} f \equiv 0$ on U together imply $\langle \lambda, f \rangle = 0$.

DEFINITION 3.4. A (N, μ)-*difference* is a difference functional which is also a (N, μ)-annihilator.

When $\mu = 0$, Definition 3.4 and the earlier one [15, Definition 3.1] are equivalent [15, Theorem 3.5] and so (N, μ)-annihilators and (N, μ)-differences are generalizations of the Nth difference studied earlier. While the main focus of this paper is the (N, μ)-difference, the first results in this section apply to the larger class of (N, μ)-annihilators.

We say that N is a *directional matrix* if:

(3.5) The elements of N lie in some open half-plane; that is; $\exists \gamma \in \mathbb{R}^d$ $\forall v \in \mathrm{N}$, $\gamma \cdot v > 0$ (or, equivalently, 0 is not in the convex hull of N) and

(3.6) N contains no distinct parallel elements.

For example, of the three matrices

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

only the last is a directional matrix.

For $\sigma \in (\mathbb{R} \backslash 0)^{\mathrm{N}}$, a vector of nonzero scalars indexed by N, define the *rescaling* $(\mathrm{N}_\sigma, \mu_\sigma)$ of (N, μ) by the rule

$$\mathrm{N}_\sigma := \{\sigma(v)\, v : v \in \mathrm{N}\}$$

and

$$(\mu_\sigma)(v) := \sigma(v)\, \mu(v).$$

Clearly, λ is a (N, μ)-difference (or annihilator) if and only if it is a $(\mathrm{N}_\sigma, \mu_\sigma)$-difference (annihilator) for any rescaling $(\mathrm{N}_\sigma, \mu_\sigma)$ of (N, μ). If N satisfies condition (3.6), and if $0 \notin \mathrm{N}$, then there exists a $\sigma \in \{-1, 1\}^{\mathrm{N}}$ for which N_σ is a directional matrix, in which case N_σ is called a *normalization* of N. (In these circumstances, σ itself may be referred to as a rescaling or normalization.)

The following proposition details a connection between the normalizations of N and the extreme points of $\mathrm{N}[0, 1]^{\mathrm{N}}$, the image under N of the unit cube in $\mathbb{R}^{\mathrm{N}}$.

PROPOSITION 3.7. *Let* $\beta \in \{0, 1\}^{\mathrm{N}}$ (*or, equivalently*, $1 - 2\beta \in \{-1, 1\}^{\mathrm{N}}$). *Then* $\mathrm{N}\beta$ *is an extreme point of* $\mathrm{N}[0, 1]^{\mathrm{N}}$ *if and only if* $1 - 2\beta$ *is a normalization of* N.

Proof. The proof consists of a series of straightforward equivalences.

$$\begin{array}{ll} & \mathrm{N}\beta \text{ is an extreme point of } \mathrm{N}[0,1]^{\mathrm{N}} \\ \text{iff} & \mathrm{N}\beta \text{ is an extreme point of } \mathrm{N}\{0,1\}^{\mathrm{N}} \\ \text{iff}^* & \mathrm{N}\beta \text{ is an extreme point of } \{\mathrm{N}\beta+(1-2\beta(v))\,v : v\in\mathrm{N}\} \\ \text{iff} & \exists\gamma\in\mathbb{R}^d \text{ such that } \forall v\in\mathrm{N},\ \gamma\cdot\mathrm{N}\beta<\gamma\cdot(\mathrm{N}\beta+(1-2\beta(v))\,v) \\ \text{iff} & \exists\gamma\in\mathbb{R}^d \text{ such that } \forall v\in\mathrm{N},\ 0<\gamma\cdot(1-2\beta(v))\,v \\ \text{iff} & 1-2\beta \text{ is a normalization of N.} \end{array}$$

The last obvious equivalence is the second (*). The forward direction is trivial. To prove the backward direction suppose that $\mathrm{N}\beta$ is extreme in $\{\mathrm{N}\beta+(1-2\beta(v))\,v : v\in\mathrm{N}\}$ so that there exists a γ in $\mathbb{R}^d$ such that $0<\gamma\cdot(1-2\beta(v))\,v$ for all v in N, and let ξ be a point in $\{0,1\}^{\mathrm{N}}$. Since ξ and β have only 0–1 entries, if $\xi(v)-\beta(v)$ is not zero then $\xi(v)-\beta(v)=1-2\beta(v)$. Therefore

$$\gamma\cdot\mathrm{N}\xi=\gamma\cdot\mathrm{N}(\xi-\beta)+\gamma\cdot\mathrm{N}\beta\leqslant\gamma\cdot\mathrm{N}\beta$$

with equality if and only if $\xi=\beta$. Consequently, $\mathrm{N}\beta$ is an extreme point of $\mathrm{N}\{0,1\}^{\mathrm{N}}$. ∎

As a simple illustration, Fig. 1 shows the image in $\mathbb{R}^2$ of the unit cube in $\mathbb{R}^3$ under the map $N=\left(\begin{smallmatrix}2&0&1\\0&2&1\end{smallmatrix}\right)$, with the points $\mathrm{N}\{0,1\}^{\mathrm{N}}$ marked with dots $\bullet$. Leaving each dot are three multiples of the columns of N, either v if $\beta(v)=0$ or $-v$ if $\beta(v)=1$. In this case, Proposition 3.7 states that the three vectors emanating from a point $\bullet$ lie in a half plane exactly when that point is extreme.

For N a directional matrix and $\mu\in\mathbb{C}^{\mathrm{N}}$, the (exponential) truncated power $T_\mu(\mathrm{N})$ is the distribution which acts on test functions ϕ by the rule [7]

$$\langle T_\mu(\mathrm{N}),\phi\rangle:=\int_{[0,\infty)^{\mathrm{N}}}e^{-\mu\cdot t}\phi(\mathrm{N}t)\,dt. \tag{3.8}$$

Condition (3.5) guarantees the existence of this integral. For completeness, $T_\mu(\varnothing)$ is defined as the Dirac δ distribution. If $\mathrm{H}\subseteq\mathrm{N}$, let $T_\mu(\mathrm{H})$ stand for $T_{\mu(\mathrm{H})}(\mathrm{H})$.

Clearly, $T_\mu(\mathrm{N})$ is supported on the cone $[\![N]\!]_+$ and

$$\forall x\in\mathbb{R}^d,\qquad T_\mu(x\mid\mathrm{N})=T_\mu(-x\mid-\mathrm{N}).$$

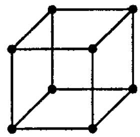

FIG. 1. A simple example for Proposition 3.7.

Among the other well-known properties of $T_\mu(\mathrm{N})$ are that if $\mathrm{H} \subseteq \mathrm{N}$ and $\mu \in \mathbb{C}^{\mathrm{N}}$ then

$$T_\mu(\mathrm{H}) * T_\mu(\mathrm{N} \backslash \mathrm{H}) = T_\mu(\mathrm{N})$$

and

$$D_{\mathrm{H},\mu} T_\mu(\mathrm{N}) = T_\mu(\mathrm{N} \backslash \mathrm{H}).$$

In particular, $D_{\mathrm{N},\mu} T_\mu(\mathrm{N}) = \delta$ so

$$D_{\mathrm{N},\mu}(T_\mu(\mathrm{N}) * \phi) = \phi \tag{3.9}$$

for any test function ϕ.

Definition 3.10. If λ is an (N, μ)-annihilator, define its *kernel* with respect to (N, μ) as the distribution

$$M(\lambda, \mathrm{N}, \mu) = \lambda * T_\mu(-\mathrm{N}).$$

In general, this convolution is well defined since λ is compactly supported [11, Sect. 5.2]. For example, if λ is a (N, μ)-difference, then for any $t \in \mathbb{R}^d$,

$$\begin{aligned} M(t \mid \lambda, \mathrm{N}, \mu) &= \lambda T_\mu(\,\cdot - t \mid \mathrm{N}) \\ &= \sum_S p_s(D)\, T_\mu(s - t \mid \mathrm{N}), \end{aligned}$$

where S and p_s are as in Eq. (3.2).

The distribution in Definition 3.10 acts as the Peano kernel of λ in the following sense. If $f \in C^\infty$ and $D_{\mathrm{N},\mu} f$ is compactly supported, then, by Eq. (3.9), f and $T_\mu(\mathrm{N}) * D_{\mathrm{N},\mu} f$ have the same (N, μ)th derivative, and therefore

$$\begin{aligned} \langle \lambda, f \rangle &= \langle \lambda, D_{\mathrm{N},\mu} f * T_\mu(\mathrm{N}) \rangle \\ &= \langle \lambda * T_\mu(-\mathrm{N}), D_{\mathrm{N},\mu} f \rangle \end{aligned} \tag{3.11}$$

for any (N, μ)-annihilator λ. (To see that these $\#\mathrm{N}$ applications of integration by parts are legitimate, note that, for any $\mathrm{H} \subset \mathrm{N}$, $\lambda * T_\mu(-\mathrm{H})$ is supported on $\operatorname{supp} \lambda - [\![\mathrm{N}]\!]_+$ and $D_{\mathrm{N},\mu} f * T_\mu(\mathrm{N} \backslash \mathrm{H})$ is supported on $\operatorname{supp} D_{\mathrm{N},\mu} f + [\![\mathrm{N}]\!]_+$, and therefore their product has compact support.)

Thus the relationship between M and λ is on par with that between the (appropriately normalized) B-spline and the corresponding difference. Another similarity between $M(\lambda, \mathrm{N}, \mu)$ and the B-spline is their compact support, detailed in the next theorem.

THEOREM 3.12. *Let* N *be a directional matrix and* $\mu \in \mathbb{C}^{\mathrm{N}}$, *and let* λ *be a* (N, μ)*-annihilator. Then* $M(\lambda, \mathrm{N}, \mu)$ *is supported on the convex hull of* $\operatorname{supp} \lambda$. *More specifically, its support lies in* $\bigcap' (\operatorname{supp} \lambda - [\![\mathrm{N}_\sigma]\!]_+)$, *where the intersection* $\bigcap'$ *is taken over all normalizations* N_σ *of* N.

This result was proven earlier in the special case that λ is an $(\mathrm{N}, 0)$-difference [15, Theorem 3.15].

Proof. For N_σ any normalization of N, the distributions $M(\lambda, \mathrm{N}, \mu)$ and $M(\lambda, \mathrm{N}_\sigma, \mu_\sigma) \prod_{\mathrm{N}} \sigma(v)$ are the same, since for any test function ϕ there exists $f \in C^\infty$ satisfying $\phi = D_{\mathrm{N},\mu} f = \prod_{\mathrm{N}} \sigma(v)\, D_{\mathrm{N}_\sigma, \mu_\sigma} f$, so that

$$\langle M(\lambda, \mathrm{N}, \mu), \phi \rangle = \langle \lambda, f \rangle = \left\langle M(\lambda, \mathrm{N}_\sigma, \mu_\sigma) \prod_{\mathrm{N}} \sigma(v), \phi \right\rangle.$$

Therefore, the support of M lies in $\operatorname{supp} \lambda + \operatorname{supp} T_{\mu_\sigma}(-\mathrm{N}_\sigma) = \operatorname{supp} \lambda - [\![\mathrm{N}_\sigma]\!]_+$.

It follows from the separation corollary that $\bigcap'(\operatorname{supp} \lambda - [\![\mathrm{N}_\sigma]\!]_+)$ is contained in the convex hull of $\operatorname{supp} \lambda$ [cf. 15, Theorem 3.15]. ∎

Since $M(\lambda, \mathrm{N}, \mu)$ has compact support, we can remove the restriction in Eq. (3.11) in which $D_{\mathrm{N},\mu} f$ has compact support. The resulting corollary is the Peano formula for (N, μ)-annihilators.

COROLLARY 3.13. *If* λ *is a* (N, μ)*-annihilator, then, for all* $f \in C^\infty(\mathbb{R}^d)$,

$$\langle \lambda, f \rangle = \langle M(\lambda, \mathrm{N}, \mu), D_{\mathrm{N},\mu} f \rangle.$$

The converse of Theorem 3.12 is straightforward to prove, and therefore (N, μ)-annihilators are characterized by having compactly supported convolutions with the associated truncated power.

THEOREM 3.14. *Let* λ *be a compactly support distribution. Then* λ *is a* (N, μ)*-annihilator if and only if* $\lambda * T_\mu(-\mathrm{N})$ *has compact support.*

Proof. If $\lambda * T_\mu(-\mathrm{N})$ has compact support, then so must $\lambda * T_\mu(-\mathrm{H})$ for any H a subset of N, since the latter distribution is simply $D_{-\mathrm{N}\setminus\mathrm{H},\mu}$ of the former. Consequently by $\#\mathrm{N}$ applications of integration by parts,

$$\langle \lambda, f \rangle = \langle \lambda * T_\mu(-\mathrm{N}), D_{\mathrm{N},\mu} f \rangle.$$

Consequently, if $D_{\mathrm{N},\mu} f$ is identically zero on the support of $\lambda * T_\mu(-\mathrm{N})$, then $\langle \lambda, f \rangle$ is zero. ∎

Let N be a directional matrix in $\mathbb{R}^d$ and S be a finite subset of $\mathbb{R}^d$. For $\mathrm{H} \subseteq \mathrm{N}$, and $s \in S$, let $(s : \mathrm{H})$ denote the set of all points in S which differ from s by an element of ran H.

THEOREM 3.15. *Let* λ *be a difference functional, let* N *be a directional matrix, and* $\mu \in \mathbb{C}^{\mathrm{N}}$. *Then the following statements are equivalent.*

A. λ *is a* (N, μ)*-difference.*

B. *For any* s *in* S, *and* $\mathscr{H}$ *a subspace of* $\mathbb{R}^d$, *let* $\mathrm{H} := \mathrm{N} \cap \mathscr{H}$. *Then* $\lambda|_{(s:\mathrm{H})}$ *is an* (H, μ)*-difference.*

C. *For any* s *in* S *and* v *in* N, $\lambda|_{(s:v)}$ *is an* $(v^{\alpha(v)}, \mu)$*-difference.*

In case $\mu = 0$, the equivalence of A and C was proven earlier [15].

Proof. $A \Rightarrow B$: This is trivial unless $\mathscr{H}$ is a proper subspace of $\mathbb{R}^d$, so suppose that to be the case. Let U be the open disk associated to λ as in Definition 3.3. If $f \in C^\infty(s + \mathscr{H})$ satisfies $D_{\mathrm{H},\mu} f \equiv 0$ on $(s + \mathscr{H}) \cap U$, then f can be extended to all of $\mathbb{R}^d$ so that $f \equiv 0$ on $S \setminus (s : \mathrm{H})$ and $D_{\mathrm{H},\mu} f$ (and therefore $D_{\mathrm{N},\mu} f$) is identically zero on U. Consequently,

$$\lambda|_{s:\mathrm{H}} f = \lambda f = 0.$$

$B \Rightarrow C$: trivial.

$C \Rightarrow A$: The proof is practically the same as one that has already appeared [15, Theorem 4.18]. ∎

COROLLARY 3.16. *Suppose the nontrivial* (N, μ)*-difference* λ *consists of function evaluations only. In the special case that* μ *is real, each* $(s : v)$ *must contain more than* $\alpha(v)$ *members. More generally, there is a positive constant* ε *depending on* N *and* μ *such that any* $(s : v)$ *which contains at most* $\alpha(v)$ *members must have diameter larger than* ε.

Whether or not μ is real, if λ consists of function evaluations only, and if s is in the support of λ, then $(s : v)$ must have more than one member.

Proof. In case $\mu=0$, this result [15, Corollary 3.3] follows from Theorem 3.15(C) via univariate polynomial interpolation. Corollary 3.16 is proven similarly using the following known proposition and corollary. ∎

PROPOSITION 3.17. *If $\mu_1, ..., \mu_n$ are real numbers, then any nontrivial element of the univariate function space*

$$\operatorname{Exp} \Pi_\mu := \ker (D-\mu_1) \cdots (D-\mu_n)$$

can have at most $n-1$ zeros, counted according to their multiplicities ($=$: CATTM). More generally, if $\mu_1, ..., \mu_n$ are complex numbers, then there exists a positive number ε such that any nontrivial element of $\operatorname{Exp} \Pi_\mu$ *can have at most $n-1$ zeros, CATTM, on any interval of length ε.*

With no direct reference for this result, we include a short proof suggested by Ron [20].

Proof. The proof in the real case appears elsewhere [19, Part 5, Problem 75]. For the complex case, suppose $t_1, ..., t_n$ are real numbers, and note that a smooth function f vanishes at $t_1, ..., t_n$ if and only if $i!\,[t_1, ..., t_i]\, f=0$ for $i=1, ..., n$.

Pick $\{u_1, ..., u_n\}$ a basis for $\operatorname{Exp} \Pi_\mu$. Then there exists a nontrivial $p \in \operatorname{Exp} \Pi_\mu$ vanishing at $t_1, ..., t_n$ if and only if the determinant

$$\det(i!\,[t_1, ..., t_i]\, u_j)_{i,j=1}^n \tag{3.18}$$

is zero.

As the points $t_1, ..., t_n$ coalesce to 0, this determinant converges to the Wronskian of $\{u_1, ..., u_n\}$ at 0, which is nonzero [5, Theorems 6.1, 6.5]. Consequently, there exists a positive ε such that, if $\{t_1, ..., t_n\} \subset (0, \varepsilon)$, then the determinant (3.18) is nonzero. That is, no nontrivial p in $\operatorname{Exp} \Pi_\mu$ can vanish at n points in $(0, \varepsilon)$. Since $\operatorname{Exp} \Pi_\mu$ is shift invariant, the proof is complete. ∎

Since $\operatorname{Exp} \Pi_\mu$ has dimension n, an immediate corollary to Proposition 3.17 is that $\operatorname{Exp} \Pi_\mu$ always allows unique Hermite-type interpolation at n points, with some restrictions if μ is nonreal.

COROLLARY 3.19. *Let f be a smooth function of one variable, and let $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$. If $\mu_1, ..., \mu_n$ are real numbers, then there exists a unique element p of* $\operatorname{Exp} \Pi_\mu$ *such that $f-p$ vanishes at $t_1, t_2, ..., t_n$, CATTM. More generally, if $\mu_1, ..., \mu_n$ are complex numbers, then there exists a positive number ε depending only on $\mu_1, ..., \mu_n$ such that, if $t_1, ..., t_n$ lie within an interval of length less than ε, then there exists a unique element p of* $\operatorname{Exp} \Pi_\mu$ *such that $f-p$ vanishes at $t_1, t_2, ..., t_n$ CATTM.*

Note that if $\operatorname{Exp} \Pi_{\mu}$ allows Hermite interpolation at any n points that are within ε of each other, then, for any positive scalar a, the space $\operatorname{Exp} \Pi_{a\mu}$ allows Hermite interpolation provided the points are within ε/a of each other.

If N is a directional matrix and $\mu \in \mathbb{C}^{\mathrm{N}}$, we shall say that μ *provides cardinal interpolation in the directions* N if, for each v in N, the $\alpha(v)$-dimensional space

$$\ker \prod_{\xi = v} (D + \mu(\xi)) \tag{3.20}$$

of univariate functions always provides an interpolant to function values at $\alpha(v)$ consecutive integers. In that case, one can always interpolate to function values at the points

$$\{v, 2v, ..., \alpha(v)\, v\} \subset \mathbb{R}^d$$

from the space

$$\operatorname{Exp} \Pi_{v^{\alpha(v)}, \mu} = \ker \prod_{\xi = v} (D_{\xi} + \mu(\xi)),$$

since, for any p in (3.21), $f(x) = p(x \cdot v/v \cdot v)$ is in $\operatorname{Exp} \Pi_{v^{\alpha(v)}, \mu}$.

The following well-known result can be proven via multivariate Bernstein polynomials [2, 18].

THEOREM 3.21. *If f is infinitely differentiable on $\mathbb{R}^d$, if K is a compact subset of $\mathbb{R}^d$, and if k is a natural number, then to any $\varepsilon > 0$ there corresponds a polynomial q such that*

$$\forall \beta \in \mathbb{Z}_+^d \text{ with } |\beta| \leqslant k, \qquad \|D^{\beta}(f - q)\|_K < \varepsilon,$$

where $\|\cdot\|_K$ is the uniform norm on K.

That is, polynomials provide simultaneous approximation to smooth functions on compact sets. For our purposes here, it will be useful to note that when $D_{\mathrm{N}, \mu} f = 0$ one can similarly approximate f with (exponential) polynomials p satisfying $D_{\mathrm{N}, \mu} p = 0$.

THEOREM 3.22. *If $f \in C^{\infty}(\mathbb{R}^d)$ satisfies $D_{\mathrm{N}, \mu} f = 0$ on a compact disk K and if $k \in \mathbb{N}$, then to any $\varepsilon > 0$ there corresponds a $q \in \operatorname{Exp} \Pi_{N, \mu}$ such that*

$$\forall \beta \in \mathbb{Z}_+^d \qquad \text{with} \quad |\beta| \leqslant k, \qquad \|D^{\beta}(f - q)\|_K < \varepsilon.$$

Proof. Without loss of generality, assume that $v \cdot v = 1$ for all v in N and that K is centered at the origin.

The proof is by induction on $\#\mathrm{N}$. For the simplest case, assume that N is the singleton set $\{v\}$ and μ is a complex scalar. Let A denote the orthogonal projection onto the hyperplane $v^{\perp}$. Since K is centered at the origin, A maps K onto a $(d-1)$-dimensional disk concentric with K. Let H be an orthonormal basis for $v^{\perp}$.

Assume $D_{v,\mu} f \equiv 0$ on K. Then for all x in K,

$$f(x) = f(Ax)\, e^{-\mu v \cdot x}.$$

Each D^{β} with $|\beta| \leqslant k$ can be expressed (uniquely) as a linear combination of

$$\{E^{\beta} := D_{v,\mu}^{\beta(v)} D_{\mathrm{H}}^{\beta(\mathrm{H})} : \beta \in \mathbb{Z}_{+}^{\mathrm{H} \cup v},\ |\beta| \leqslant k\}.$$

If q is any polynomial, then

$$q(Ax)\, e^{-\mu v \cdot x} \in \operatorname{Exp} \Pi_{v,\mu},$$

and

$$E^{\beta}(f(x) - q(Ax)\, e^{-\mu v \cdot x}) = (D_{v,\mu}^{\beta(v)} e^{-\mu v \cdot x})(D_{\mathrm{H}}^{\beta(\mathrm{H})}(f(Ax) - q(Ax))).$$

By Theorem 3.21, the norm of this on K can be made arbitrarily small for all $|\beta| \leqslant k$ by choosing the polynomial q appropriately.

For the inductive step, choose $v \in \mathrm{N}$, and let A and $\{E^{\beta}\}$ be as above. For any $p \in \Pi$ and $q \in \operatorname{Exp} \Pi_{\mathrm{N} \backslash v, \mu}$, define the exponential polynomial P by the rule

$$P(x) = p(Ax)\, e^{-\mu v \cdot x} + e^{-\mu v \cdot x} \int_{Ax}^{x} e^{\mu v \cdot y} q(y)\, dy,$$

where $\int$ is a line integral. It is straightforward to see that

$$E^{\beta} p(Ax)\, e^{-\mu v \cdot x} = (D_{\mathrm{H}}^{\beta(\mathrm{H})} p(Ax))\, D_{v,\mu}^{\beta(v)} e^{-\mu v \cdot x}$$

which is zero if $\beta(v) > 0$, and

$$D_{v,\mu} e^{-\mu v \cdot x} \int_{Ax}^{x} e^{\mu v \cdot y} q(y)\, dy = q(y),$$

so $D_{\mathrm{N},\mu} P = 0$. Also, from the identities

$$\forall \eta \in \mathrm{H}, \qquad D_{\eta} e^{-\mu v \cdot x} \int_{Ax}^{x} e^{\mu v \cdot y} q(y)\, dy = e^{-\mu v \cdot x} \int_{Ax}^{x} e^{\mu v \cdot y} D_{\eta} q(y)\, dy$$

and

$$e^{\mu\nu\cdot x}f(x)=f(Ax)+\int_{Ax}^{x} e^{\mu\nu\cdot y}D_{\nu,\mu}f(y)\,dy$$

it follows that, for every multiindex β with $|\beta|\leqslant k$,

$$\begin{aligned}E^{\beta}(f(x)-P(x))&=E^{\beta}(f(Ax)-p(Ax))\,e^{-\mu\nu\cdot x}\\&\quad+E^{\beta}e^{-\mu\nu\cdot x}\int_{Ax}^{x} e^{\mu\nu\cdot y}(D_{\nu,\mu}f(y)-q(y))\,dy\\&=D_{\mathrm{H}}^{\beta(\mathrm{H})}(f(Ax)-p(Ax))\,D_{\nu,\mu}^{\beta(\nu)}e^{-\mu\nu\cdot x}\\&\quad+D_{\nu,\mu}^{\beta(\nu)}e^{-\mu\nu\cdot x}\int_{Ax}^{x} e^{\mu\nu\cdot y}D_{\mathrm{H}}^{\beta(\mathrm{H})}(D_{\nu,\mu}f(y)-q(y))\,dy.\end{aligned}$$

By Theorem 3.21, the first term on the right side can be make arbitrarily small by choice of the polynomial p, while, by the inductive hypothesis, the second can be made arbitrarily small by choice of $q\in\operatorname{Exp}\Pi_{\mathrm{N}\backslash\nu,\mu}$, completing the proof. ▮

Theorem 3.22 has three relatively immediate corollaries.

COROLLARY 3.23. *For* N *is directional matrix, k a natural number, and $\mu\in\mathbb{C}^{\mathrm{N}}$, to every $f\in C^{\infty}(\mathbb{R}^d)$ with $D_{\mathrm{N},\mu}f=0$ on a disk U containing the finite set S, there corresponds a $p\in\operatorname{Exp}\Pi_{\mathrm{N},\mu}$ such that $D^{\beta}(f-p)(s)=0$ for all $s\in S$ and $\beta\in\mathbb{Z}_+^d$ with $|\beta|\leqslant k$.*

COROLLARY 3.24. *The difference functional λ is a (N,μ)-difference if and only if it annihilates* $\operatorname{Exp}\Pi_{\mathrm{N},\mu}$.

COROLLARY 3.25. *Let* N, μ, S, *and* k *be as in Corollary* 3.23. *For each $s\in S$, let Π_s be a finite-dimensional space of d-variate polynomials, and let $\Pi_s(D)$ be the associated space of constant-coefficient differential operators. Let f be a smooth function. Then it is possible to find an element p of* $\operatorname{Exp}\Pi_{\mathrm{N},\mu}$ *such that $\Pi_s(D)(f-p)(s)=0$ for all $s\in S$ if and only if f is annihilated by every (N,μ)-difference λ of the form*

$$\lambda\colon f\mapsto\sum_{S}p_s(D)\,f(s)$$

with $p_s\in\Pi_s$ for each s.

Proof. Let K be a compact disk containing S. All three corollaries follow from the finite dimensionality of the image of $C^k(K)$ under the map

$$p \mapsto \{D^\beta p(s) : |\beta| \leqslant k, s \in S\}$$

(which maps functions to real vectors indexed by the set $\{(\beta, s) : |\beta| \leqslant k, s \in S\}$). Since $\operatorname{Exp} \Pi_{\mathrm{N}, \mu}$ is dense in $\ker D_{\mathrm{N}, \mu}$ in the topology of $C^k(K)$, these two spaces have the same image under this map, proving Corollary 3.23. Corollary 3.24 follows immediately.

Let $\operatorname{Exp} \Pi_{\mathrm{N}, \mu}(S, k)$ denote this image. Corollary 3.25 follows from Corollary 3.24 and the observation that

$$\operatorname{Exp} \Pi_{\mathrm{N}, \mu}(S, k)^{\perp\perp} = \operatorname{Exp} \Pi_{\mathrm{N}, \mu}(S, k). \quad \blacksquare$$

As a simple example, suppose $\{x_0, ..., x_n\}$ and $\{y_0, ..., y_m\}$ are increasing sequences of real numbers and

$$S := \{(x_i, y_j) : i \in \{0..n\}, j \in \{0..m\}\}.$$

Let N consist of n copies of the vector $(1, 0)$ and m copies of $(0, 1)$, and let $\mu = 0$. Then it is possible to interpolate to the values of a function f on S by some $p \in \Pi$ satisfying $D_{\mathrm{N}} p = 0$ if and only if $\lambda f = 0$ for λ any linear combination of function evaluations on S which annihilates $\ker D_{\mathrm{N}}$. As it turns out [15, Proof of Theorem 4.1], any such λ is a scalar multiple of the tensor product divided difference. Therefore, in this simple case, Corollary 3.25 reduces to the well-known fact that one can find a polynomial p agreeing with f on S for which

$$\left(\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial y}\right)^m p = 0$$

if and only if $[x_0, ..., x_n][y_0, ..., y_m] f(x, y) = 0$.

4. SPANNING SETS FOR (N, μ)-DIFFERENCES

We begin this section by an immediate extension of an earlier result.

THEOREM 4.1. *Let* N *be a directional matrix whose distinct columns are linearly independent, and let $\mu \in \mathbb{C}^{\mathrm{N}}$. Then any (N, μ)-difference consisting entirely of function evaluations can be written, after a linear change of variables, as a linear combination of tensor product (N, μ)-dividend differences.*

Proof. We sketch the proof, since it is essentially the same as one published earlier [15, Theorem 4.1]. See that paper to review the concept of a tensor product divided difference. A review of exponential divided differences can be found in a more recent paper [17, Sect. 3].

It will suffice to prove Theorem 4.1 in case the unique elements of N form the standard orthonormal basis for $\mathbb{R}^d$.

Add points to S as necessary so that it forms a tensor product grid (that is, the Cartesian product of d finite subsets of $\mathbb{R}$). Add more points to S as necessary so as to guarantee (by Corollary 3.19) the existence of tensor product (N, μ)-dividend differences (consisting of function evaluations only) supported on S. Repeat the following step until impossible:

Choose a maximal element s^* of S (in the lexicographic order). Modify λ by subtracting from it a scalar multiple of a tensor product (N, μ)-divided difference so that

$$s^* \notin \text{new } S := \text{supp new } \lambda \subset \text{old } S.$$

This process terminates when, for lack of the requisite number of points in some direction, there are no more tensor product (N, μ)-divided differences supported on S. Corollary 3.16 now implies that no nontrivial (N, μ)-difference consisting of function evaluations only can be supported on the resulting S, making the resulting λ the zero functional. ▮

The remaining question of interest is what might serve for a spanning set of all (N, μ)-differences if the distinct columns of N are linearly dependent. We give a partial solution to that problem in this section, showing that, with the additional restriction that $S \subset \mathbb{Q}^d$, a (N, μ)-difference which consists entirely of function evaluations must be a linear combination of forward differences. Several definitions and technicalities will precede these main results.

Define the forward difference operators (which send functions to functions) by the rules

$$\nabla_\mu^\nu \colon f \mapsto f - e^{\mu(\nu)} f(\cdot - \nu)$$

and

$$\nabla_\mu^{\mathrm{N}} := \prod_{\nu \in \mathrm{N}} \nabla_\mu^\nu.$$

The corresponding *forward differences* (which send functions to scalars) are defined as follows. Let

$$\delta_{0,\mu}^\nu \colon f \mapsto f(0) - e^{\mu(\nu)} f(\nu).$$

As a distribution, $\delta_{0,\mu}^\nu = \nabla_\mu^\nu \delta$, so that $\nabla_\mu^\nu f = \delta_{0,\mu}^\nu * f$. Similarly define

$$\delta_{0,\mu}^{\mathrm{N}} := \nabla_\mu^{\mathrm{N}} \delta \tag{4.2}$$

so that $\nabla_\mu^{\mathrm{N}} f = \delta_{0,\mu}^{\mathrm{N}} * f$. For arbitrary t in $\mathbb{R}^d$, define

$$\delta_{t,\mu}^{\mathrm{N}}\colon f \mapsto \delta_{0,\mu}^{\mathrm{N}} f(\cdot + t).$$

It is straightforward to see from Eq. (4.2) that if $\sigma \in \{-1, 1\}^{\mathrm{N}}$,

$$\delta_{0,\mu}^{\mathrm{N}} = \left(\prod_{v \in \mathrm{M}} (-e^{\mu(v)}) \right) \delta_{\mathrm{M}\mathbb{1},\mu_\sigma}^{\mathrm{N}_\sigma}, \tag{4.3}$$

where $M := \{v \in \mathrm{N}\colon \sigma(v) = -1\}$. That is, a forward difference in the directions N beginning from 0 is, up to a scalar factor, the same as a forward difference in the rescaled directions N_σ, beginning at another point, $\mathrm{M}\mathbb{1}$.

In case $\mu = 0$, the forward difference $\delta_{t,\mu}^{\mathrm{N}}$ may be written more simply as δ_t^{N}. For completeness, define

$$\delta_{t,\mu}^{\varnothing}\colon f \mapsto f(t).$$

Definition 4.4. Let N satisfy condition (3.5). For x and y in $\mathbb{R}^d$, we write $x \overset{\mathrm{N}}{\leqslant} y$ provided that there exists $z \in \mathbb{R}_+^{\mathrm{N}}$ such that $x + \mathrm{N}z = y$. Furthermore, we write $x \overset{\mathrm{N}}{<} y$ if $x \overset{\mathrm{N}}{\leqslant} y \neq x$.

It follows from condition (3.5) that $\overset{\mathrm{N}}{\leqslant}$ is a partial order.

Proposition 4.5. *Let* N *be a directional matrix. If S and T are finite subsets of $\mathbb{R}^d$ and if there exist nonzero scalars $a(s)$ for all $s \in S$ and $b(t)$ for all $t \in T$ such that*

$$\sum_S a(s)\, \delta_s^{\varnothing} = \sum_T b(t)\, \delta_{t,\mu}^{\mathrm{N}} \tag{4.6}$$

then the $\overset{\mathrm{N}}{\leqslant}$-wise minimal elements of T belong to S.

Proof. Expanding each $\delta_{t,\mu}^{\mathrm{N}}$,

$$\sum_T b(t)\, \delta_{t,\mu}^{\mathrm{N}} = \sum_T b(t) \sum_{\mathrm{M} \subseteq \mathrm{N}} c(\mathrm{M})\, \delta_{t+\mathrm{M}}^{\varnothing},$$

where $c(\varnothing)$ is a nonzero scalar. If t^* is $\overset{\mathrm{N}}{\leqslant}$-minimal in T, then the only difference $\delta_{t,\mu}^{\mathrm{N}}$ having support at t^* is $\delta_{t^*,\mu}^{\mathrm{N}}$; hence t^* is in S and $a(t^*) = b(t^*)\, c(\varnothing)$. ∎

In general it is not necessary that $T \subset S$ for Eq. (4.6) to be true. For instance, if $\mathrm{N} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\lambda = \delta_{(1,1),0}^{\mathrm{N}} - \delta_{(0,0),0}^{\mathrm{N}}$, then the point $(1, 1)$ is not in the support of λ.

More detailed information on the relation between T and S is provided by the next corollary.

COROLLARY 4.7. *Under the same hypotheses as Proposition* 4.5, *the extreme points of* $T+\mathrm{N}\{0,1\}^{\mathrm{N}}$ *must belong to* S.

Proof. If $t^*+\mathrm{N}\beta$ is an extreme point in $T+\mathrm{N}\{0,1\}^{\mathrm{N}}$ for some t^* in T and β in $\{0,1\}^{\mathrm{N}}$, then $\mathrm{N}\beta$ is extreme in $\mathrm{N}\{0,1\}^{\mathrm{N}}$. As in Proposition 3.7, let $\sigma=1-2\beta$ be the associated normalization of N. By (4.3), the functional (4.6) can be rewritten as

$$\sum_T b(t)\,\delta^{\mathrm{N}}_{t,\mu}=\sum_T c(t)\,\delta^{\mathrm{N}\sigma}_{t+\mathrm{N}\beta,\mu} \tag{4.8}$$

for some nonzero scalars $\{c(t) : t\in T\}$.

Since $t^*+\mathrm{N}\beta$ is an extreme point in $T+\mathrm{N}\{0,1\}^{\mathrm{N}}$, there exists $\gamma\in\mathbb{R}^d$ such that, for any $s+\mathrm{N}\xi$ in $T+\mathrm{N}\{0,1\}^{\mathrm{N}}$ other than $t^*+\mathrm{N}\beta$,

$$\gamma\cdot(t^*+\mathrm{N}\beta)<\gamma\cdot(s+\mathrm{N}\xi). \tag{4.9}$$

In particular, letting $s=t^*$ and $\xi=\beta+(1-2\beta(v))\,v$ implies that

$$\gamma\cdot(1-2\beta(v))\,v>0$$

for any v in N. Therefore, if $s+\mathrm{N}\beta\overset{\mathrm{N}_\sigma}{\leqslant}t^*+\mathrm{N}\beta$ for s in T, then $\gamma\cdot(s+\mathrm{N}\beta)\leqslant\gamma\cdot(t^*+\mathrm{N}\beta)$ so that Eq. (4.9) implies $s=t^*$. Consequently, $t^*+\mathrm{N}\beta$ is $\overset{\mathrm{N}_\sigma}{\leqslant}$-minimal in $T+\mathrm{N}\beta$ and, in light of Eq. (4.8), Proposition 4.5 implies that $t^*+\mathrm{N}\beta$ lies in S. ▮

LEMMA 4.10. *Let* N *be a finite subset of* $\mathbb{Z}^d$ *and suppose the set* L *is covered by the union of finitely many disjoint shifts of the lattice generated by* N; *that is,*

$$L\subseteq\bigcup_{i=1}^{k}(z_i+\mathrm{N}\mathbb{Z}^{\mathrm{N}})$$

for some points $z_1, z_2, ..., z_k$ *in* $\mathbb{R}^d$. *Suppose further that*

$$v\in\mathrm{N},\qquad kv\in L-L\Rightarrow k\in\mathbb{N}.$$

Then, for λ *any* (N,μ)*-difference supported on* L, *the restriction of* λ *to any* $z_i+\mathrm{N}\mathbb{Z}^{\mathrm{N}}$ *is also a* (N,μ)*-difference.*

Proof. By Theorem 3.15, it will suffice to show that, for any $i\in\{1..k\}$, any $s\in\operatorname{supp}\lambda$, and any $v\in\mathrm{N}$, the restriction of λ to $(s:v)\cap(z_i+\mathrm{N}\mathbb{Z}^{\mathrm{N}})$ is an $(v^{\alpha(v)},\mu)$-difference, where $(s:v)$ is the set of points in $\operatorname{supp}\lambda$ that differ from s by a multiple of v. This is trivially the case if $(s:v)\cap(z_i+\mathrm{N}\mathbb{Z}^{\mathrm{N}})$ is empty, so suppose that for some scalar k, the point $s+kv$ is simultaneously in supp λ and $z_i + \mathrm{N}\mathbb{Z}^{\mathrm{N}}$. Since this means $kv \in L - L$, the scalar k must be an integer and $s \in z_i + \mathrm{N}\mathbb{Z}^{\mathrm{N}}$.

For any $t \in (s : v)$, there exists a scalar k, necessarily an integer, for which $t - s = kv$, and therefore $t \in z_i + \mathrm{N}\mathbb{Z}^{\mathrm{N}}$. It follows that

$$(s : v) = (s : v) \cap (z_i + \mathrm{N}\mathbb{Z}^{\mathrm{N}})$$

and, since Theorem 3.16 implies that the restriction of λ to $(s : v)$ is a $(v^{\alpha(v)}, \mu)$-difference, the proof is complete. ▮

The next two results, the main ones of this section, form another characterization of (N, μ)-differences. Clearly, if (N^*, μ^*) is a rescaling of (N, μ), then any linear combination of (N^*, μ^*) forward differences is a (N, μ)-difference. As it turns out (Corollary 4.13), there are no other (N, μ)-differences which depend solely on function values at rational points.

Theorem 4.11. *Assume that* N *is a rational directional matrix for which*

$$v \in \mathrm{N}, \qquad kv \in \mathrm{N}\mathbb{Z}^{\mathrm{N}} \Rightarrow k \in \mathbb{Z} \tag{4.12}$$

and that μ *provides cardinal interpolation in the directions* N. *If* λ *is a* (N, μ)*-difference consisting entirely of function-evaluations at points in* $\mathrm{N}\mathbb{Z}^{\mathrm{N}}$, *then there exist* $T \subset \mathbb{Z}^{\mathrm{N}}$ *and* $b \in \mathbb{R}^T$ *such that*

$$\lambda = \sum_T b(t)\, \delta^{\mathrm{N}}_{\mathrm{N}t, \mu}.$$

While the hypotheses of Theorem 4.11 may seem somewhat restrictive, we will see in the following corollary that a similar conclusion holds for a broader class of (N, μ)-differences.

Corollary 4.13. *Let* N *be a directional matrix and let* λ *be a* (N, μ)*-difference consisting entirely of function evaluations at rational points* S. *Then there exists a rescaling* $(\mathrm{N}_\sigma, \mu_\sigma)$ *of* (N, μ), *a finite subset* T *of* $\mathbb{R}^d$, *and a vector* $b \in \mathbb{R}^T$ *such that*

$$\lambda = \sum_T b(t)\, \delta^{\mathrm{N}_\sigma}_{t, \mu_\sigma}.$$

Proof of Corollary 4.13. The conclusion is obvious if λ is identically zero. If λ is nontrivial, then, since it consists of function evaluations only, Corollary 3.16 implies that for any v in N there exists a multiple of v in $S - S$. By choosing the appropriate rescaling, we may assume N is this collection of rational directions. For every v in N, define v_ε to be the smallest positive multiple of v found in the (necessarily discrete) lattice $\mathrm{N}\mathbb{Z}^{\mathrm{N}}$. Define N_ε to be the set of all resulting v_ε and let μ_ε be the corresponding rescaling of μ. Then the lattice $\mathrm{N}_\varepsilon\mathbb{Z}^{\mathrm{N}}$ generated by N_ε is the same as that generated by N.

It will suffice to prove Corollary 4.13 in case S lies in ran N, since the restriction of λ to any $(s : \mathrm{N})$ is a (N, μ)-difference. Therefore, we may assume that for each s in S there is a rational vector $t \in \mathbb{Q}^{\mathrm{N}}$ such that $\mathrm{N}_\varepsilon t = s$. Choose a natural number m so that $mt \in \mathbb{Z}^{\mathrm{N}}$ for every such t, and define $\mathrm{N}_\omega := \mathrm{N}_\varepsilon/m$ and $\mu_\omega := \mu_\varepsilon/m$. Finally, pick a natural number ℓ (guaranteed by the remarks following Corollary 3.19) so that $\mu_\sigma := \mu_\omega/\ell$ provides cardinal interpolation in the directions $\mathrm{N}_\sigma := \mathrm{N}_\omega/\ell$. Then S lies in the lattice generated by N_σ, and N_σ satisfies condition (4.12). Therefore, by Theorem 4.11, λ is a linear combination of various shifts of $\delta^{\mathrm{N}_\sigma}_{0,\mu_\sigma}$. ∎

Proof of Theorem 4.11. We first show that it will suffice to prove Theorem 4.11 under the additional assumption that N is an integer matrix and contains a nonzero multiple of each of the columns $e_1, e_2, ..., e_d$ of the $d \times d$ identity matrix.

Suppose Theorem 4.11 were true under these additional assumptions, and assume that $\mathrm{N} \subset \mathbb{Q}^d$ has rank $c \leqslant d$ and satisfies the hypotheses of the theorem.

Take H a basis in N. Then there exists a rational $c \times d$ matrix K such that KH is the $(c \times c)$ identity. Consequently, $\mathrm{HKH}t = \mathrm{H}t$ for any $t \in \mathbb{R}^{\mathrm{H}}$. That is, the restriction of HK to ran N is the identity.

Choose a natural number m so that $\Gamma := m\mathrm{KN}$ is an integer matrix. The typical element γ of Γ is $m\mathrm{K}v$, and, since H is a subset of N, there can be found in Γ multiples of $e_1, e_2, ..., e_c$. If $k\gamma \in \Gamma\mathbb{Z}^\Gamma$ for some $\gamma = m\mathrm{K}v \in \Gamma$ and real number k, then multiplication on the left by H implies that $kv \in \mathrm{N}\mathbb{Z}^{\mathrm{N}}$, and, by hypothesis, this forces k to be a natural number. Therefore Γ satisfies the hypotheses of Theorem 4.11.

Assume λ is a (N, μ)-difference consisting entirely of function evaluations on $\mathrm{N}\mathbb{Z}^{\mathrm{N}}$. Define

$$\xi\colon f \mapsto \lambda(f \circ m\mathrm{K}).$$

Then ξ is a linear combination of function evaluations on $\Gamma\mathbb{Z}^\Gamma$.

Since N and Γ are in one-to-one correspondence, one can view the given μ as a member of $\mathbb{C}^\Gamma$. For any smooth function f,

$$(D_{\Gamma,\mu} f) \circ m\mathrm{K} = D_{\mathrm{N},\mu}(f \circ m\mathrm{K}), \tag{4.14}$$

and therefore λ being a (N, μ)-difference forces ξ to be a (Γ, μ)-difference. Therefore, Theorem 4.11, in the special case that the directional matrix is integral and contains multiples of $e_1, \ldots, e_c$, implies the existence of a set $T \subset \mathbb{Z}^\Gamma$ such that

$$\xi = \sum_T b(t)\, \delta^\Gamma_{\Gamma t, \mu}.$$

It is not hard to see from (4.2) that

$$\delta^\Gamma_{\Gamma t, \mu}(f \circ m^{-1}\mathrm{H}) = \delta^{\mathrm{N}}_{\mathrm{N}t, \mu} f \tag{4.15}$$

so that, for any function f,

$$\begin{aligned} \lambda f &= \xi(f \circ m^{-1}\mathrm{H}) \\ &= \sum_T b(t)\, \delta^{\mathrm{N}}_{\mathrm{N}t, \mu} f. \end{aligned}$$

With this sufficiency established, assume for the remainder of the proof that N is an integer matrix containing elements parallel to each of $e_1, \ldots, e_d$.

Partition N into the classes

$$\mathrm{N}_i = \mathrm{N} \cap (\operatorname{ran}(e_i, \ldots, e_d) \backslash \operatorname{ran}(e_{i+1}, \ldots, e_d)).$$

For v in N, let $v\!\downarrow\, := v(i)$ where i is the smallest integer for which $v(i) \neq 0$. Since the conclusion of Theorem 4.11 is independent of the normalization of N, we may assume that each v in N has been multiplied by ± 1 so as to achieve $v\!\downarrow\, > 0$. This is a normalization of N, as is any choice of ± 1 that causes

$$\forall i\, \forall v, \eta \in \mathrm{N}_i, \qquad \operatorname{sign}(v\!\downarrow) = \operatorname{sign}(\eta\!\downarrow), \tag{4.16}$$

since $\mathrm{N}a = 0$ and $a \geqslant 0$ then implies $a = 0$ (cf. (3.5)).

Choose $\gamma \in \mathbb{R}^d$ so that $\gamma \cdot \mathrm{N} > 0$ and $\theta \in \mathbb{Z}^d$ so that the support S of λ lies in $\theta + \mathbb{Z}^d_+$. Then modify λ and S according to the following algorithm.

Algorithm 4.17. Define

$$S' := \{s \in S : \operatorname{supp} \delta^{-\mathrm{N}}_s \subset \theta + \mathbb{Z}^d_+\}.$$

If S' is nonempty, choose $s \in S'$ so that $\gamma \cdot s \geqslant \gamma \cdot S'$ and c a scalar so that $\lambda - c\delta^{-\mathrm{N}}_{s, -\mu}$ has no support at s. Redefine $\lambda := \lambda - c\delta^{-\mathrm{N}}_{s, -\mu}$ and $S := \operatorname{supp} \lambda$. Repeat until $S' = \varnothing$.

Proposition 4.18. *Algorithm* 4.17 *stops after finitely many iterations.*

Proof. Since N contains a positive multiple of each e_i for each i between 1 and d, each component $\gamma(i)$ of γ is positive. Consequently, for any number r, the set

$$\{z \in \theta + \mathbb{Z}_+^d : \gamma \cdot z = r\}$$

has finite cardinality. At any time during Algorithm 4.17, S' is a subset of $\theta + \mathbb{Z}_+^d$, so at no time is $\max \gamma \cdot S < \gamma \cdot \theta$.

At each iteration, a point s at which $\gamma \cdot$ takes its maximum on S' is removed, and the points x added to S', if any, satisfy

$$\gamma \cdot x < \gamma \cdot s.$$

Were the loop to repeat indefinitely, $\max \gamma \cdot S'$ would eventually be reduced to less than $\gamma \cdot \theta$, a contradiction. ∎

Continuing with the proof of Theorem 4.11, since $S' = \varnothing$ when Algorithm 4.17 ends, S lies in

$$\mathrm{N}\mathbb{Z}^{\mathrm{N}} \cap (\theta + (\mathbb{Z}_+^d \setminus \mathbb{Z}')),$$

where

$$\mathbb{Z}' := \{z \in \mathbb{Z}^d : \forall \mathrm{K} \subseteq \mathrm{N},\ z - \mathrm{K}\mathbb{1} \in \mathbb{Z}_+^d\}. \tag{4.19}$$

To better understand S, we make the following observations.

PROPOSITION 4.20. *If u is a minimal element of $\mathbb{Z}'$ in the sense of the standard $\leqslant$, then to each i in $\{1..d\}$, there corresponds a $\mathrm{K} \subset \mathrm{N}$ for which $(u - \mathrm{K}\mathbb{1})(i) = 0$.*

Proof. Define the multiinteger w by

$$w(i) := \min_{\mathrm{K} \subseteq \mathrm{N}} (u - \mathrm{K}\mathbb{1})(i) \geqslant 0.$$

For any $\mathrm{H} \subseteq \mathrm{N}$ and i in $\{1..d\}$,

$$(u - w - \mathrm{H}\mathbb{1})(i) = (u - \mathrm{H}\mathbb{1})(i) - \min_{\mathrm{K} \subseteq \mathrm{N}} (u - \mathrm{K}\mathbb{1})(i) \geqslant 0.$$

Consequently, $u - w \in \mathbb{Z}'$, and since u is minimal, $w = 0$, proving the proposition. ∎

PROPOSITION 4.21. *There is a unique minimal element of $\mathbb{Z}'$.*

Proof. That a minimal element exists follows from $\mathbb{Z}' \subseteq \mathbb{Z}^d_+$.

Suppose both u and v are minimal in $\mathbb{Z}'$ and therefore not comparable. Then $u(i) < v(i)$ for some i in $\{1..d\}$. By Proposition 4.20, there is a subset K of N for which

$$0 = (v - \mathrm{K}\mathbb{1})(i) > (u - \mathrm{K}\mathbb{1})(i),$$

contradicting $u \in \mathbb{Z}'$ and proving the proposition. ∎

It is an easy consequences of Definition (4.19) that if $u \in \mathbb{Z}'$, then any multiinteger greater or equal u must also belong to $\mathbb{Z}'$. That is, $\mathbb{Z}' = \{u..\infty\}$ where u denotes the minimal element of $\mathbb{Z}'$. Therefore, since the support S of λ is finite, there exists a multiinteger β such that

$$S \subset \mathrm{N}\mathbb{Z}^{\mathrm{N}} \cap \{\theta..\beta\} \backslash \{\theta + u..\infty\}.$$

The proof of Theorem 4.11 is completed by the following result.

THEOREM 4.22. *Let* N *be a directional integer matrix with* $v\!\downarrow\, > 0$ *for each* v *in* N. *Assume also that* μ *provides cardinal interpolation in the directions* N. *Assume that* N *contains a multiple of* e_i *for each* i *in* $\{1..d\}$, *and that* N *satisfies* (4.12). *Let* θ *and* β *be in* $\mathbb{Z}^d$ *and let* u *denote the minimal element of* $\mathbb{Z}'$. *Then the only* (N, μ)*-difference* λ *supported on the set*

$$S = \mathrm{N}\mathbb{Z}^{\mathrm{N}} \cap \{\theta..\beta\} \backslash \{\theta + u..\infty\}$$

is the trivial $\lambda = 0$.

Proof. The proof is by induction on d and $\#\mathrm{N}$.

In case $d = 1$, the hypothesis (4.12) forces $\mathrm{N} = v^{\alpha(v)}$, the set consisting of $\alpha(v)$ copies of the positive integer v. In this case, the minimal element of

$$\mathbb{Z}' = \{z \geqslant 0 : z - \alpha(v)\, v \geqslant 0\}$$

is simply $u = \alpha(v)\, v$. The set S is at most the set of all integer multiples of v between $\theta \in \mathbb{Z}$ and $\theta + \alpha(v)\, v - 1$. Since one can always interpolate to function values on this set from $\operatorname{Exp} \Pi_{v^{\alpha(v)}, \mu}$, the only $(v^{\alpha(v)}, \mu)$-difference supported on S is the trivial one.

Assume now that $d > 1$ and that Theorem 4.22 (and consequently Theorem 4.11) holds for matrices $\tilde{\mathrm{N}} \subset \mathbb{Z}^{\tilde{d}}$ and integers $\tilde{d}$ provided either

$$\tilde{d} < d \tag{4.23}$$

or

$$\tilde{d} = d \qquad \text{and} \qquad \#\tilde{\mathrm{N}} < \#\mathrm{N}. \tag{4.24}$$

Take β minimal so that the (N, μ)-difference λ is supported on

$$S = \mathrm{N}\mathbb{Z}^{\mathrm{N}} \cap \{\theta..\beta\} \backslash \{\theta + u..\infty\}.$$

We will first show that $\beta(1)$ is necessarily less than $\theta(1) + u(1)$. Toward that end, assume

$$\beta(1) \geqslant \theta(1) + u(1). \tag{4.25}$$

Define

$$\begin{aligned} \mathrm{M} &:= \mathrm{N} \cap \mathrm{ran}\{e_2, ..., e_d\} \\ \mathrm{K} &:= (\mathrm{N}\backslash\mathrm{M})\backslash\mathrm{ran}\{e_1\} \\ \mathrm{H} &:= \mathrm{N}\backslash\mathrm{ran}\{e_1\} \\ &= \mathrm{K} \cup \mathrm{M}. \end{aligned}$$

By hypothesis, H is a proper subset of N.

Define w to be the unique minimal element of

$$\{z \in \mathbb{Z}^d : \forall \mathrm{X} \subseteq \mathrm{M},\, z - \mathrm{X}\mathbb{1} \in \mathbb{Z}^d_+\}$$

so that for all i in $\{1..d\}$ and all $\mathrm{X} \subseteq \mathrm{M}$,

$$(w - \mathrm{X}\mathbb{1})(i) \geqslant 0 \tag{4.26}$$

and for every i there exists an X for which $(w - \mathrm{X}\mathbb{1})(i) = 0$. Because $\mathrm{M} \subseteq \mathrm{ran}\{e_2, ..., e_d\}$, the minimality of w forces $w(1) = 0$.

Since β is minimal, choose s in S satisfying $s(1) = \beta(1)$.

Define θ' to have $\beta(1)$ for its first coordinate and, for its 2nd, ..., dth coordinates, the same as $\theta + u - w$.

We will first show that λ has no support at s if $s \ngeqslant \theta'$. This is vacuously true in case $\mathrm{K} = \varnothing$, since then $u(\{2..d\}) = w(\{2..d\})$, so assume $\mathrm{K} \neq \varnothing$ and $s \ngeqslant \theta'$.

Since $s(1) = \theta'(1)$, assume $\exists i^* > 1$ such that

$$s(i^*) < \theta(i^*) + u(i^*) - w(i^*).$$

By Proposition 4.20, there is a subset X of H for which

$$(u - \mathrm{X}\mathbb{1})(i^*) = 0.$$

By Eq. (4.26),

$$(\theta + u - (\mathrm{X} \cap \mathrm{M}\, \mathbb{1}))(i^*) \geqslant (\theta + u - w)(i^*) > s(i^*),$$

so that

$$(s-(\mathrm{X}\cap\mathrm{K})\,\mathbb{1})(i^*)=(s-\mathrm{X}\mathbb{1}+(\mathrm{X}\cap\mathrm{M})\,\mathbb{1})(i^*)$$
$$<(\theta+u-\mathrm{X}\mathbb{1})(i^*)=\theta(i^*).$$

Let $\hat{\mathrm{H}}=(-\mathrm{K})\cup\mathrm{M}$, a normalization of H of the form (4.16), and let $\hat{\mu}$ be the corresponding normalization of μ. Since λ is a (N,μ)-difference, it is also an $(\hat{\mathrm{H}},\hat{\mu})$-difference, and since $\mathrm{N}\mathbb{Z}^{\mathrm{N}}$ is the union of finitely many disjoint shifts of $\hat{\mathrm{H}}\mathbb{Z}^{\hat{\mathrm{H}}}$, Lemma 4.10 and the induction hypothesis (4.24) imply that λ is a linear combination of $(\hat{\mathrm{H}},\hat{\mu})$-forward differences. That is, for some $V\subset\mathrm{N}\mathbb{Z}^{\mathrm{N}}$,

$$\lambda=\sum_V c(v)\,\delta^{\hat{\mathrm{H}}}_{v,\hat{\mu}}.$$

Since $-\mathrm{K}\downarrow<0$, the point s cannot lie in the support of $\delta^{\hat{\mathrm{H}}}_{v,\hat{\mu}}$ if $v(1)<\beta(1)$. If $v(1)=\beta(1)$ and $s\in\operatorname{supp}\delta^{\hat{\mathrm{H}}}_{v,\hat{\mu}}$, then $s=v+\mathrm{P}\mathbb{1}$ for some $\mathrm{P}\subseteq\mathrm{M}$. But then

$$(v-(\mathrm{X}\cap\mathrm{K})\,\mathbb{1}+\mathrm{P}\mathbb{1})(i^*)=(s-(\mathrm{X}\cap\mathrm{K})\,\mathbb{1})(i^*)<\theta(i^*).$$

Thus $V+\hat{\mathrm{H}}\{0,1\}^{\hat{\mathrm{H}}}$ contains a point which is not $\geqslant\theta$. Therefore, this set contains an extreme point which is not $\geqslant\theta$. But this contradicts Corollary 4.7, which implies that the extreme points of $V+\hat{\mathrm{H}}\{0,1\}^{\hat{\mathrm{H}}}$ must lie in S.

Hence if $s\in S$ satisfies $s(1)=\beta(1)$ and $s\not\geqslant\theta'$, then s cannot be in the support of λ.

The restriction λ' of λ to the hyperplane $x(1)=\beta(1)$ is therefore supported on the lattice points in $\mathrm{N}\mathbb{Z}^{\mathrm{N}}$ that also lie in

$$\{\theta'..\beta\}\backslash\{\theta+u..\infty\}=\{\theta'..\beta\}\backslash\{\theta'+w..\infty\}.$$

Since these lattice points are covered by finitely many shifts of $\mathrm{M}\mathbb{Z}^{\mathrm{M}}$, Lemma 4.10 and the induction hypothesis (4.23) imply that λ' is identically zero.

Having ruled out the alternative (4.25), we now conclude that

$$\beta(1)<\theta(1)+u(1).$$

Therefore the support of λ lies in

$$\mathrm{N}\mathbb{Z}^{\mathrm{N}}\cap\{\theta..\beta\}.$$

Let $\mathrm{H} := \mathrm{N} \backslash \operatorname{ran} e_d$, by hypothesis a proper subset of N. By Proposition 4.20, there exists $\mathrm{M} \subseteq \mathrm{H}$ for which

$$(u - \mathrm{M}\mathbb{1})(1) = 0.$$

Since λ is an (H, μ)-difference, the induction hypothesis implies that it can be written as a linear combination of forward differences:

$$\lambda = \sum_V c(v)\, \delta^{\mathrm{H}}_{v,\mu}.$$

By Corollary 4.7, the extreme points of $V + \mathrm{H}\{0, 1\}^{\mathrm{H}}$ must lie, as S does, in the set

$$\{x \in \mathbb{R}^d : \theta(1) \leqslant x(1) \leqslant \beta(1)\}.$$

But if $v(1) \geqslant \theta(1)$, then

$$(v + \mathrm{M}\mathbb{1})(1) \geqslant (\theta + \mathrm{M}\mathbb{1})(1) = \theta(1) + u(1) > \beta(1)$$

which implies that $V + \mathrm{H}\{0, 1\}^{\mathrm{H}}$ contains an extreme point x with $x(1) > \beta(1)$. Therefore the finite set V must have no extreme points, which forces it to be empty and λ to be identically zero, which completes the proofs of Theorems 4.22 and 4.11. ∎

5. APPLICATIONS AND OPEN QUESTIONS

The results of the last two sections have applications in the areas of box splines and polynomial interpolation.

The first of these follows from Theorems 3.14 and 4.11.

COROLLARY 5.1. *Let B be a linear combination of finitely many shifts of the truncated power $T_\mu(\mathrm{N})$ by rational points; that is, for some finite $S \subset \mathbb{Q}^d$ and $c \in \mathbb{R}^S$,*

$$B = \sum_{s \in S} c(s)\, T_\mu(\cdot - s \mid \mathrm{N}).$$

Then B is compactly supported if and only if it is a linear combination of finitely many shifts of a box spline $B_{\mu^*}(\mathrm{N}^*)$ for some rescaling (N^*, μ^*) of (N, μ).

By Corollary 3.25, one can restate the conclusion of Theorem 4.22 in terms of polynomial interpolation as follows.

COROLLARY 5.2. *Under the same hypotheses as Theorem* 4.22, *for any smooth function f there exists $p \in \operatorname{Exp} \Pi_{\mathrm{N},\mu}$ such that $p(s) = f(s)$ for every s in S.*

One familiar example of a set S satisfying the hypotheses of Theorem 4.22 is gained by removing an extreme point from the support of the forward difference functional. Specifically, suppose that N satisfies the hypotheses of Theorem 4.22 and define

$$S := \{-\mathrm{K}\mathbb{1} : \varnothing \neq \mathrm{K} \subseteq \mathrm{N}\} \subset \mathrm{N}\mathbb{Z}^{\mathrm{N}}.$$

By Eq. (4.19), $S \geqslant -u$, where u is the minimal element of $\mathbb{Z}'$ and since $\mathrm{N}\downarrow > 0$, no point in S is in $\mathbb{Z}^d_+$. Therefore the finite set S satisfies

$$S \subset \mathrm{N}\mathbb{Z}^{\mathrm{N}} \cap \{-u..\beta\} \backslash \{0..\infty\}$$

for some multiinteger β. Since $\operatorname{Exp} \Pi_{\mathrm{N},\mu}$ is translation invariant, this proves the following corollary.

COROLLARY 5.3. *Let* N *and* μ *satisfy the hypothesis of Theorem* 4.22. *For any smooth function f there exists $p \in \operatorname{Exp} \Pi_{\mathrm{N},\mu}$ such that p agrees with f at the points*

$$\{\mathrm{K}\mathbb{1} : \mathrm{K} \subset \mathrm{N}, \mathrm{K} \neq \mathrm{N}\}.$$

Since $\mathrm{N}\mathbb{1}$ is an extreme point among all the points in the support of the forward difference $\delta^{\mathrm{N}}_{0,\mu}$, the conclusion of Corollary 5.3 is equivalent to saying that there exists p in $\operatorname{Exp} \Pi_{\mathrm{N},\mu}$ agreeing with f on $\{\mathrm{K}\mathbb{1} : \mathrm{K} \subseteq \mathrm{N}\}$ if and only if $\delta^{\mathrm{N}}_{0,\mu} f = 0$.

This result is easily extended in the next corollary.

COROLLARY 5.4. *Let* N *be a directional rational matrix and let μ provide cardinal interpolation in the directions* N. *Suppose that* N *satisfies* (4.12). *Then for any smooth function f there exists $p \in \operatorname{Exp} \Pi_{\mathrm{N},\mu}$ agreeing with f on $\{\mathrm{K}\mathbb{1} : \mathrm{K} \subseteq \mathrm{N}\}$ if and only if $\delta^{\mathrm{N}}_{0,\mu} f = 0$.*

Proof. The proof consists of showing that assumptions placed on N in Corollary 5.3 can be relaxed without forfeiting the conclusion.

As a first step, suppose N and μ satisfy all the hypotheses of Theorem 4.22 and that $(\mathrm{N}_\sigma, \mu_\sigma)$ is a renormalization of (N, μ). Let $M = \{v \in \mathrm{N} : \sigma(v) = -1\}$. By (4.3), there exists a function p in

$$\operatorname{Exp} \Pi_{\mathrm{N},\mu} = \operatorname{Exp} \Pi_{\mathrm{N}_\sigma, \mu_\sigma}$$

agreeing with f on

$$\{\mathrm{K}\mathbb{1} : \mathrm{K} \subseteq \mathrm{N}\} = \{\mathrm{M}\mathbb{1} + \mathrm{K}\mathbb{1} : \mathrm{K} \subseteq \mathrm{N}_\sigma\}$$

if and only if

$$\delta^{\mathrm{N}}_{0,\mu} f = 0 = \delta^{\mathrm{N}_\sigma}_{\mathrm{M}\mathbb{1},\mu_\sigma} f.$$

The requirement that N be normalized so as to ensure $\mathrm{N}\downarrow > 0$ is therefore superfluous: given that N satisfies all the other requirements of Theorem 4.22, the conclusion of Corollary 5.3 holds for any normalization of N.

To finish the proof, assume that $\mathrm{N} \subset \mathbb{Q}^d$ has rank $c \leqslant d$ and that

$$v \in \mathrm{N}, \qquad kv \in \mathrm{N}\mathbb{Z}^{\mathrm{N}} \Rightarrow k \in \mathbb{Z}.$$

Pick $\mathrm{H} \subseteq \mathrm{N}$ a basis for ran N. Then there exists a $c \times d$ rational matrix K such that $\mathrm{KH} = I$, and, as a result, $\mathrm{HKH}t = \mathrm{H}t$ for any t in $\mathbb{R}^{\mathrm{H}}$. That is, the restriction of HK to ran N is the identity. Pick an integer m so that the $c \times \#\mathrm{N}$ matrix $\Gamma := m\mathrm{KN} = \{m\mathrm{K}v : v \in \mathrm{N}\}$ is integral. Since $m\mathrm{KH}$ is contained in Γ, the latter contains multiples of each of $e_1, ..., e_c$. Furthermore, if $k\gamma \in \Gamma\mathbb{Z}^\Gamma$ for some $\gamma \in \Gamma$ and $k \in \mathbb{R}$, then $k\mathrm{H}\gamma \in m\mathrm{HKN}\mathbb{Z}^\Gamma$. By its definition, Γ is in one-to-one correspondence with N, and therefore one can view $\mathbb{Z}^\Gamma$ and $\mathbb{Z}^{\mathrm{N}}$ as the same. This plus that fact that $\mathrm{HK}v = v$ for all $v \in \mathrm{N}$ imply that $k\mathrm{H}\gamma \in m\mathrm{N}\mathbb{Z}^{\mathrm{N}}$. If γ is written as $m\mathrm{K}v$, it follows that $kv \in \mathrm{N}\mathbb{Z}^{\mathrm{N}}$, which implies that k must be an integer.

If f is a function with domain $\mathbb{R}^d$, then $f \circ m^{-1}\mathrm{H}$ has domain $\mathbb{R}^c$, and, by Corollary 5.3 and the first paragraph of this proof, there exists

$$p \in \operatorname{Exp} \Pi_{\Gamma,\mu} \tag{5.5}$$

such that

$$p = f \circ m^{-1}\mathrm{H} \text{ on } \{M\mathbb{1} : M \subseteq \Gamma\} \tag{5.6}$$

if and only if

$$\delta^{\Gamma}_{0,\mu}(f \circ m^{-1}\mathrm{H}) = 0. \tag{5.7}$$

But, by (4.14), Eq. (5.5) is equivalent to

$$p \circ m\mathrm{K} \in \operatorname{Exp} \Pi_{\mathrm{N},\mu}$$

and, since $\mathrm{HKN} = \mathrm{N}$, (5.6) is equivalent to

$$p \circ m\mathrm{K} = f \qquad \text{on} \quad \{M\mathbb{1} : M \subseteq \mathrm{N}\}$$

and, by (4.15), (5.7) is equivalent to

$$\delta^{\mathrm{N}}_{0,\mu} f = 0,$$

completing the proof. ▮

Finally, we mention two open questions in this general area.

Examples indicate that the conclusion of Corollary 5.4 is true in more general circumstances than the hypotheses describe. For instance, it is not hard to show that if $\mathrm{N} = \{u, v, w\}$ is a directional matrix, then one can interpolate to f at $\{\mathrm{K}\mathbb{1} : \mathrm{K} \subseteq \mathrm{N}\}$ from Exp $\Pi_{\mathrm{N},0}$ if and only if $\delta^{\mathrm{N}}_{0,0} f = 0$, whether or not N is rational or satisfies condition (4.12). On the other hand, if N consists of 24 copies each of (2, 0), (0, 5), and (3, 3), a matrix which violates condition (4.12), the conclusion of Corollary 5.4 is false. Can Corollary 5.4 be generalized by weakening its hypotheses?

Do Theorems 4.1 and 4.11 have a mutual extension? In each, the conclusion is that a (N, μ)-difference which consists of function evaluations only can be written as a linear combination of convolutions of not necessarily evenly spaced divided differences in independent directions (Theorem 4.1) or evenly spaced divided differences in possibly dependent directions (Theorem 4.11). Under what more general hypothesis is it true that every (N, μ)-difference is a linear combination of convolutions of not necessarily evenly spaced divided differences in possibly dependent directions? Such differences arise in the study of the box-like spline introduced earlier [16, 17]. Might these differences serve as a spanning set for all (N, μ)-differences of function values only?

Without further restrictions, the answer appears doubtful, as suggested by the following bivariate example. Define a difference λ by the rule

$$\begin{aligned}\lambda\colon f \mapsto f(\pi+1, \pi+1) - f(\pi, \pi+1) - f(\pi+1, 1)\\ + f(\pi, 0) + f(0, 1) - f(0, 0).\end{aligned}$$

By Theorem 3.16, λ is an $(\mathrm{N}, 0)$-difference, where

$$\mathrm{N} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Since each v in N has multiplicity one, the differences associated to the corresponding box-like splines are simply forward differences in the directions

$$\mathrm{N}_\sigma = \begin{pmatrix} a & 0 & c \\ 0 & b & c \end{pmatrix}$$

with a, b, and c nonzero reals. In this case, the above question reduces to the following. Is

$$\lambda = \sum_T d(t)\, \delta_t^{\mathrm{N}_{\sigma(t)}}$$

for some multiset T in $\mathbb{R}^2$ and scalars $d(t)$ and rescalings $\sigma(t)$ depending on $t \in T$?

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